Two Partial Orders for Littlewood-Richardson Tableaux

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Abstract: In this manuscript we show that two partial orders defined on the set of Littlewood-Richardson fillings of shape $\beta \setminus \gamma$ and content α are equivalent if $\beta \setminus \gamma$ is a horizontal and vertical strip. In fact, we give two proofs for the equivalence of the box order and the dominance order for fillings. Both are algorithmic. The first of these proofs emphasizes links to the Bruhat order for the symmetric group and the second provides a more straightforward construction of box moves. This work is motivated by the known result that the equivalence of the two combinatorial orders leads to a description of the geometry of the representation space of invariant subspaces of nilpotent linear operators.

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1. Introduction

Let α, β, γ be partitions. By $\mathcal{T}^{\beta}_{\alpha,\gamma}$ we denote the set of all Littlewood-Richardson fillings (LR-fillings) of shape $\beta \setminus \gamma$ and content α. In Section 2 we define two partial orders $\leq_{\mathsf{box}}$ and $\leq_{\mathsf{dom}}$ on the set $\mathcal{T}^{\beta}_{\alpha,\gamma}$. These orders are defined combinatorially and are of importance in the theory of invariant subspaces of nilpotent linear operators. They control the geometry of varieties of invariant subspaces of nilpotent linear operators, as they describe the degeneration relation and the boundaries of the irreducible components, see [4, 5, 6]. Therefore, it is important to investigate properties of these orders. One of the main results of the paper is the following theorem.

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Theorem 1.1. *Let X, Z be LR-fillings of the same shape and the same content. If the shape is a horizontal and vertical strip then the following conditions are equivalent.*

1. $Z \leq_{\mathsf{box}} X$,
2. $Z \leq_{\mathsf{dom}} X$.

It is easy to see that in general, the box relation implies the dominance relation (see Lemma 2.1). For the converse, Example 2 in Section 6.2 shows that the condition that $\beta \setminus \gamma$ be a vertical strip is necessary. We show in [6] that several other relations of geometric or of algebraic nature lie between the box and the dominance relations. If those two are equal, then all the relations coincide.

We present two different proofs of Theorem 1.1. Both proofs are constructive. The first one, presented in Section 4, shows connections of our problem with the Bruhat order in the symmetric group S_n where $n = |\alpha|$.

The second proof, given in Section 5, gives a more straightforward algorithm. Given LR-fillings such that $Z \leq_{\mathsf{dom}} X$, both algorithms compute a sequence of box moves that convert X to Z. This proves that $Z \leq_{\mathsf{box}} X$.

Finally, in Section 6, we describe some properties of the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$. We prove that there exists exactly one minimal and exactly one maximal element, and that all saturated chains have the same length.

2. Definitions and notation

Following [2, 8] we recall definitions and notation connected with LR-fillings.

Notation: Recall that a *partition* $\alpha = (\alpha_1, \ldots, \alpha_s)$ is a finite non-increasing sequence of natural numbers; we picture α by its Young diagram which consists of s rows of lengths given by the parts of α. The *transpose* α' of α is given by the formula

$$\alpha'_j = \#\{i : \alpha_i \geq j\},$$

it is pictured by the transpose of the Young diagram for α. Two partitions α, $\widetilde{\alpha}$ are in the *dominance partial order,* in symbols $\alpha \leq_{\mathsf{dom}} \widetilde{\alpha}$, if the inequality

$$\alpha_1 + \cdots + \alpha_j \leq \widetilde{\alpha}_1 + \cdots + \widetilde{\alpha}_j$$

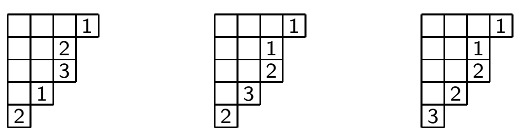
holds for each j.

Fix two partitions $\gamma \subseteq \beta$ such that the Young diagram for γ is contained in the Young diagram for β. The skew diagram $\beta \setminus \gamma$ is said to be a *vertical strip* if $\beta_i \leq \gamma_i + 1$ holds for all i, and a *horizontal strip* if $\beta' \setminus \gamma'$ is a vertical strip. A *rook strip* is a horizontal and vertical strip.

Given three partitions $\alpha = (\alpha_1, \ldots, \alpha_s)$, β, γ, we will consider fillings of $\beta \setminus \gamma$ which have α_1 entries 1's, α_2 entries 2's, etc. We describe such a filling as having the *content* α and the *shape* $\beta \setminus \gamma$. The *type* of the filling, (α, β, γ), records the content and shape. A filling is said to be an *LR-filling* if the following three conditions are satisfied:

- in each row, the entries are weakly increasing,
- in each column, the entries are strictly increasing,
- (*lattice permutation property*) for each $u > 1$ and for each column c: on the right hand side of c, the number of entries $u - 1$ is at least the number of entries u.

Example: Let $\alpha = (2, 2, 1)$, $\beta = (4, 3, 3, 2, 1)$, $\gamma = (3, 2, 2, 1)$. We have to fill the skew diagram $\beta \setminus \gamma$ with two $\boxed{1}$'s, two $\boxed{2}$'s, and one $\boxed{3}$. Due to the conditions on an LR-filling, this can be done in exactly three ways.



In this example, $\beta \setminus \gamma$ is a vertical but not a horizontal strip.

Notation: One can represent an LR-filling X by a sequence of partitions

$$X = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$$

where s is the number of rows of α and $\gamma^{(i)}$ denotes the region in the Young diagram β which contains the entries $\boxed{}$, $\boxed{1}$, ..., $\boxed{i}$. If X has type (α, β, γ), then $\gamma = \gamma^{(0)}$, $\beta = \gamma^{(s)}$, and $\alpha_i = |\gamma^{(i)} \setminus \gamma^{(i-1)}|$ for $i = 1, \ldots, s$.

In the example above, the first filling is given by the sequence of partitions $X = [(3, 2, 2, 1), (4, 2, 2, 2), (4, 3, 2, 2, 1), (4, 3, 3, 2, 1)]$.

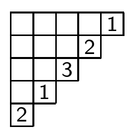
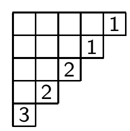
We introduce two partial orders on the set $\mathcal{T}^{\beta}_{\alpha,\gamma}$ of all LR-fillings of the same type.

Definition: Two LR-fillings $Z = [\delta^{(0)}, \dots, \delta^{(s)}]$, $X = [\gamma^{(0)}, \dots, \gamma^{(s)}]$ of the same type are *related in the dominance partial order,* in symbols $Z \leq_{\mathsf{dom}} X$, if for each i, the corresponding partitions $\delta^{(i)}$, $\gamma^{(i)}$ are related in the dominance partial order, i.e. $\delta^{(i)} \leq_{\mathsf{dom}} \gamma^{(i)}$.

In the example above, the first tableau is smallest in dominance order, followed by the second, followed by the third.

Definition: Suppose X, Z are LR-fillings of the same type which we assume to be a horizontal strip or a vertical strip. We say Z is obtained from X by *a decreasing box move* if after two entries in X have been exchanged in such a way that the smaller entry is in the lower position after the exchange, we obtain Z by re-sorting the entries in each row and in each column if necessary. We denote by $\leq_{\mathsf{box}}$ the partial order generated by box moves.

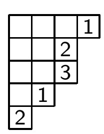
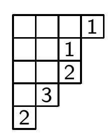
Here is an example:

Z :  $<_{\mathsf{box}}$ X : 

To get Z we apply to X the following sequence of moves. First we exchange $\boxed{2}$ and $\boxed{3}$ in rows 4 and 5, then $\boxed{1}$ and $\boxed{2}$ in rows 2 and 3 and finally $\boxed{1}$ and $\boxed{3}$ in rows 3 and 4.

Note that here the short sequence of box moves (given by exchanging $\boxed{1}$ and $\boxed{2}$ in rows 2 and 4, and $\boxed{2}$ and $\boxed{3}$ in rows 3 and 5) is not admissible due to the lattice permutation property.

In the next example in X we exchange $\boxed{1}$ and $\boxed{3}$ in rows 2 and 4 and sort entries in the third column.

Z :  $<_{\mathsf{box}}$ X : 

We finish this section by establishing the following fact.

LEMMA 2.1. *For LR-fillings of the same type, the $\leq_{\mathsf{box}}$-order implies the $\leq_{\mathsf{dom}}$- order.*

Proof. Suppose that the LR-filling $Z = [\delta^{(0)}, \dots, \delta^{(s)}]$ is obtained from $X = [\gamma^{(0)}, \dots, \gamma^{(s)}]$ by a decreasing box move based on swapping entries a and b with, say, $a < b$. The process of reordering the entries in each row or column will not affect entries less than a or larger than b, so the partitions

$\gamma^{(0)}, \ldots, \gamma^{(a-1)}$, and $\gamma^{(b)}, \ldots, \gamma^{(s)}$ remain unchanged. The partitions $\delta^{(\ell)}$, $\gamma^{(\ell)}$ for $a \leq \ell < b$ are different and satisfy $\delta^{(\ell)} <_{\mathsf{dom}} \gamma^{(\ell)}$ (since the defining partial sums can only increase). This shows that $Z <_{\mathsf{dom}} X$. □

3. Motivation

Our investigation of the partial orders for LR-fillings is motivated by an application to short exact sequences of nilpotent linear operators. Let k be an algebraically closed field. A nilpotent k-linear operator $N = (V, T)$ consists of a finite dimensional k-vector space V together with a nilpotent k-linear map $T : V \to V$. Such an operator $N_\alpha = (V, T)$ is given uniquely, up to isomorphy, by a partition α recording the sizes of the Jordan blocks of the action of T on the vector space V. Given two nilpotent linear operators $N = (V, T)$ and $N' = (V', T')$, a morphism from N to N' is a k-linear map $\phi : V \to V'$ such that $T'\phi = \phi T$.

The Green-Klein Theorem [3] establishes the link with LR-fillings:

Theorem 3.1. *For partitions α, β, γ, there exists a short exact sequence $0 \to N_\alpha \to N_\beta \to N_\gamma \to 0$ of nilpotent linear operators if and only if there is an LR-filling of type $(\alpha', \beta', \gamma')$.* □

More precisely, if A is the image of the embedding $N_\alpha \to N_\beta = B$ in the short exact sequence, then the tableau $X = [\gamma^{(0)}, \ldots, \gamma^{(s)}]$ of the sequence is given by $s = \min\{\ell : T^\ell A = 0\}$, and the transposes of the partitions $\gamma^{(\ell)}$ are given by the Jordan types of the action of T on the factors $B/T^\ell A \cong N_{(\gamma^{(\ell)})'}$.

Together, the $k[T]$-monomorphisms in the short exact sequences form a constructible subset $\mathbb{V}_{\alpha,\gamma}^\beta$ of the affine variety $\mathrm{Hom}_k(N_\alpha, N_\beta)$. Note that each irreducible component $\overline{\mathbb{V}}_X$ of $\mathbb{V}_{\alpha,\gamma}^\beta$ is given as the closure of the set of sequences with corresponding LR-filling X. All irreducible components have the same dimension.

Definition: Two LR-tableaux X, Z of the same type are said to be in *boundary relation*, $Z \preccurlyeq_{\mathsf{boundary}} X$, if $\mathbb{V}_X \cap \overline{\mathbb{V}}_Z \neq \emptyset$ holds.

The following theorem is shown in [6]:

Theorem 3.2. *Suppose X, Y are LR-tableaux of the same type and of shape which is a rook strip. If Y is obtained from X by a decreasing box move, then $Y \prec_{\mathsf{boundary}} X$.* □

More precisely, given X, Y in box relation, we construct a one-parameter family of embeddings $M(\lambda)$, and for each embedding a short exact sequence $0 \to L \to M(\lambda) \to N \to 0$, such that the following properties are satisfied:

1. $L \oplus N$ has tableau X;
2. the sequence is split exact if $\lambda = 0$;
3. $M(\lambda)$ has tableau Y if $\lambda \neq 0$.

Thus, the above result provides a link between the combinatorial relation given by box moves, the algebraic relation given by short exact sequences and the geometric boundary relation.

In general, the boundary relation implies the dominance relation ([6]). Hence, the transitive closure $\leq_{\mathsf{boundary}}$ of the boundary relation $\preccurlyeq_{\mathsf{boundary}}$ is a partial order.

We obtain from Theorem 3.2 the following chain of implications for tableaux X, Z of the same type such that the box relation is defined:

$$Z \leq_{\mathsf{box}} X \quad \text{implies} \quad Z \leq_{\mathsf{boundary}} X \quad \text{implies} \quad Z \leq_{\mathsf{dom}} X$$

As a consequence, Theorem 1.1 yields the following result:

Theorem 3.3. *The following statements are equivalent for LR-tableaux X, Z of the same type and of shape which is a rook strip.*

1. $Z \leq_{\mathsf{box}} X$
2. $Z \leq_{\mathsf{boundary}} X$
3. $Z \leq_{\mathsf{dom}} X$ □

The case where the partition α has at most two parts is particularly well understood since then for each shape $\beta \setminus \gamma$, there are only finitely many isomorphism classes of short exact sequences in $\mathbb{V}^{\beta}_{\alpha,\gamma}$. In this situation, the boundary relation has a combinatorial description in terms of arc diagrams, see [4, Theorem 1.2], and is, in fact, transitive and equivalent to several algebraic and combinatorial relations, in particular to $\leq_{\mathsf{box}}$ and $\leq_{\mathsf{dom}}$.

4. The Bruhat order and the first proof of the main result

Notation: Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a partition of n, $\beta \setminus \gamma$ a skew diagram which is a rook strip, and X a filling with content α. We denote by $\pi(X)$ the standardization of the word obtained from reading X from top right to bottom left, defined as follows: We read the filling from top right to bottom left, and we read the 1's as (in order) 1 up to α_1, we read the 2's as $\alpha_1 + 1$ to $\alpha_1 + \alpha_2$, and so forth.

Example: Let $\beta = (5, 4, 3, 2, 1)$, $\gamma = (4, 3, 2, 1)$ and $\alpha = (2, 2, 1)$. Consider the following fillings of $\beta \setminus \gamma$:

$$Z: \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 2 \\ \cline{1-4} \ & \ & 3 \\ \cline{1-3} \ & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \quad , \quad X: \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & 1 \\ \hline \ & \ & \ & 1 \\ \cline{1-4} \ & \ & 2 \\ \cline{1-3} \ & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

Note that $\pi(Z) = (1, 3, 5, 2, 4)$ and $\pi(X) = (1, 2, 3, 4, 5)$.

For any filling X with content α, the permutation $\pi(X)$ will satisfy that the numbers $1, 2, \ldots, \alpha_1$ are in increasing order, the numbers $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$ are in increasing order, and so forth. We denote the set of such permutations by S^α. The map π is a bijection between the set of fillings of given shape $\beta \setminus \gamma$ having content α and S^α.

Proposition 4.1. *Let X, Z be fillings of $\beta \setminus \gamma$ of the content α. The relation $Z <_{\mathsf{dom}} X$ holds if and only if $\pi(X) < \pi(Z)$ in the Bruhat order.*

Proof. Note that $x \leq z$ in the Bruhat order in S_n if and only if for all i, the increasing rearrangement of the first i letters of x has each letter less than or equal to the corresponding letter in the rearrangement of the first i letters of z. (This is [1, Theorem 2.6.3].) The condition is equivalent to the following. For each i and each j, the number of elements less than or equal to j in the first i letters of x is greater than or equal to the corresponding number for z.

Let $x = \pi(X)$, $z = \pi(Z)$ and suppose $x \leq z$ in the Bruhat order. Let $1 \leq i \leq n$, $1 \leq r \leq s$. If we consider $j = \alpha_1 + \cdots + \alpha_r$, we see that $x \leq z$ implies that the number of boxes labeled $1, \ldots, r$ in the first i boxes of X is greater than or equal to the corresponding number in Z. This implies that $Z \leq_{\mathsf{dom}} X$.

On the other hand, if $Z \leq_{\mathsf{dom}} X$, then by the i-th box of X, there have been at least as many boxes with entry 1 as in Z, at least as many boxes

with entry 1 or 2 in X as in Z, and so on. This implies that the increasing rearrangement of the first i entries of x has each letter less than or equal the corresponding entry of the increasing rearrangement of the first i entries of z. Finally $x \leq z$ in the Bruhat order. $\square$

Now we consider the "box move". We recall that a decreasing box move on X swaps two entries of X, so that the larger entry moves higher, and the smaller entry moves lower. We have already established, in Lemma 2.1, that applying a decreasing box move moves us down in dominance order. We wish to prove the converse result, that if $X > Z$ in dominance order, then there is a sequence of decreasing box moves which take us from X to Z, and such that at each intermediate step, the filling stays lattice.

Notation: For any $i = 1, \dots, n-1$, denote by $s_i = (i, i+1) \in S_n$ the adjacent transposition. Let $x \in S_n$ and let $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ with k minimal. The number k is called the *length* of x and we denote it by $\ell(x) = k$.

For $\pi \in S_n$, define the *inversions* of π to be pairs $a < b$ such that $\pi^{-1}(a) > \pi^{-1}(b)$.

We now state several elementary lemmas.

LEMMA 4.2. *The inversions of π are the transpositions t such that $\ell(t\pi) < \ell(\pi)$.*

Proof. Left-multiplying by $t = (a\ b)$ has the effect of swapping a and b in the one-line notation for π. This decreases the length if and only if the larger number comes before the smaller, that is, $\pi^{-1}(a) > \pi^{-1}(b)$. $\square$

LEMMA 4.3. *$x \in S^\alpha$ if and only if $\ell(s_i x) > \ell(x)$ for all $i \notin \{\alpha_1, \alpha_2 + \alpha_1, \dots\}$.*

Proof. Note that $x = (x_1, \dots, x_n) \in S^\alpha$ if and only if for all $i \notin \{\alpha_1, \alpha_1 + \alpha_2, \dots\}$, we have that i appears to the left of $i+1$. The effect of left-multiplying x by s_i is to swap the positions of i and $i+1$. This increases the length if and only if i started to the left of $i+1$. $\square$

Thanks to the previous lemma, we recognize S^α as a parabolic quotient. It consists of the minimal-length coset representatives for $S_\alpha \backslash S_n$, where S_α permutes separately the entries $1, \dots, \alpha_1$, the entries $\alpha_1 + 1, \dots, \alpha_1 + \alpha_2$, and so on.

Notation: For $x \in S_n$, define the α-recording tableau of x to be the filling with entries $x^{-1}(1), \dots, x^{-1}(\alpha_1)$ in the first row, $x^{-1}(\alpha_1 + 1), \dots, x^{-1}(\alpha_1 + \alpha_2)$

in the second row, etc. A tableau of size n is said to be *standard* if it contains each of the numbers from 1 to n exactly once, and its entries increase along rows and down columns.

Example: Let $x = (1, 3, 5, 2, 4)$ and $\alpha = (2, 2, 1)$. The α-recording tableau of x is the following:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Let $L \subseteq S_n$ be the set of permutations which correspond to LR-fillings of content α.

Lemma 4.4. *For $x \in S_n$, we have that $x \in L$ if and only if the α-recording tableau for x is standard.*

Proof. The row-increasing condition for standardness is equivalent to $x \in S^\alpha$. Suppose this is satisfied, and let X be the corresponding tableau. The number of entries $\leq t$ in row i of the recording tableau is equal to the number of boxes in X labeled i we have seen in reading the first t entries of X. Denote this number by $\lambda_i^{(t)}$. The lattice condition is then equivalent to the condition that for each t, the resulting shape $(\lambda_1^{(t)}, \lambda_2^{(t)}, \ldots)$ is a partition, and this condition is easily seen to be equivalent to column-increasingness. $\square$

We can also think of a standard α-recording tableau as the record of a process of adding a sequence of single boxes, resulting in the final shape α, such that at each step, the shape is a partition.

Lemma 4.5. *If $x \lessdot z$ is a cover in Bruhat order, such that x and z correspond to lattice fillings X and Z respectively, then there is a decreasing box move from X to Z.*

Proof. A cover in Bruhat order swaps two entries, and moves the larger number up. $\square$

By Lemma 4.5, our desired result follows if we show that, if $x < z$ in Bruhat order, then there is a sequence of Bruhat covers from x to z which remains in the set L. It is therefore enough to establish the following proposition.

Proposition 4.6. *If $x < z$ in Bruhat order, with $x, z \in L$, then there is a cover $x \lessdot y$ with $y \in L$ such that $y \leq z$.*

Proof. Write an expression for z as a product of adjacent transpositions $s_{i_1}\cdots s_{i_p}$. Since $x < z$, there is a subword of this word which equals x. As in the proof of [1, Lemma 2.2.1], choose one such that the leftmost omitted reflection is as far to the right as possible (i.e. $x = s_{i_1}\cdots\widehat{s}_{i_{j_1}}\cdots\widehat{s}_{i_{j_q}}\cdots s_{i_p}$ with $j_1 < \cdots < j_q$ such that j_1 is maximal, and $\widehat{s}$ means that s is omitted). Define y to be obtained from this subword by adding back in the leftmost simple reflection in the word for z not in the chosen subword for x (i.e. $y = s_{i_1}\cdots\widehat{s}_{j_2}\cdots\widehat{s}_{j_q}\cdots s_{i_p}$).

By the proof of [1, Lemma 2.2.1], this is a reduced expression for a permutation which covers x in Bruhat order and lies below z. Note that our conventions differ from those of [1, Lemma 2.2.1] by reversing left and right.

The fact that $y \in S^\alpha$ follows from the proof of [1, Theorem 2.5.5], because we have identified S^α as a certain parabolic quotient. (Note that in [1], they generally take parabolic quotients with the subgroup acting on the right. Our reversal mentioned above is consistent with the fact that we take parabolic quotients with the subgroup acting on the left.) The proof of [1, Theorem 2.5.5] uses the same construction as the previously-cited [1, Lemma 2.2.1], it just proves an additional property under a stronger hypothesis.

We now verify that $y \in L$. We can write $y = xt$ for t a transposition. Suppose $t = (a\ b)$ with $a < b$. This has the effect that x and y take different values restricted to $\{a, b\}$. Thus, x^{-1} and y^{-1} differ when restricted to $\{x(a), x(b)\}$, on which the values of x^{-1} and y^{-1} are a, b. This means that the tableaux for x and y differ in that a and b swap places. The fact that $\ell(x) < \ell(y)$ means that, as we read the tableau in usual order (left to right and top to bottom), we will encounter first a, then b in x, and first b then a in y.

We refer to the box of the α-recording tableau for x which contains the entry a as A, and the box which contains the entry b as B. We fix this nomenclature, so that even if we refer to a different tableau of shape α, the boxes A and B are the same (while now perhaps containing different values). We can thus say that in the tableau for y, A contains the value b, and B contains the value a. By what we have already shown, B is in a row weakly below the row of A.

We think of the α-recording tableau of x as determining a sequence of boxes which we add, starting at the empty shape, and eventually reaching the shape α. We add the boxes in numerical order (according to the entry they are filled with). As we have pointed out already, the fact that the boxes can be added in this order such that at each step, the shape is a partition,

captures the condition of corresponding to a standard tableau, and thus of coming from an element of L. For this, see the proof of Lemma 4.4.

So, to establish the lattice condition for y, we must establish that if we modify the order of adding boxes determined by x, so that we add B when we would have added A, and vice versa, each of the intermediate shapes will still be a partition.

Suppose the tableau for x looks as below, where we have specified only the entries equal to a and b. As we have already established, the tableau for y is the same except that a and b change places.

			a	+
			+	
	−			
−	b			

Since the tableau corresponding to x is standard, the entries in boxes marked with a $+$ must be greater than a, and the entries in boxes marked with a $-$ must be less than b. There are two ways that the tableau corresponding to y could fail to be standard: if one of the entries in a $+$ box were less than or equal to b, or if one of the entries in a $-$ box were greater than or equal to a.

The fact that $\ell(y) = \ell(x) + 1$ implies that the entries between a and b in one-line notation for x are all either greater than b or less than a [1, Lemma 2.1.4]. This means that between adding A and B in the sequence of partitions for x, we will only add boxes which are either (i) in a higher row than A, (ii) in the same row as, but to the left of A, (iii) in a row below B, or (iv) to the right of B in the same row. Of these, (ii) is impossible because we are at the point where we can add A, so all the boxes to the left of it in the same row must already have been added. Also, (iv) is impossible because, in adding the boxes determined by x, we cannot add any boxes to the right of B until we have added B. Thus, the boxes which will be added between adding A and adding B are all in rows of the α-recording tableau either above the row of A or below the row of B. This means that, as we construct the x tableau, the boxes marked $+$ and $-$ will not be touched between when we add A and when we add B. Since the boxes marked $+$ were empty when we added A, they are still empty when we added B, and thus they contain entries at least equal to b. Since the boxes marked $-$ are filled when we add B, they must have been filled when we added A, so they are filled with entries less than or equal to a.

Two bad possibilities remain: that the boxes A and B are actually ad-

jacent, either in the same row, or in the same column. We must rule these possibilities out as well.

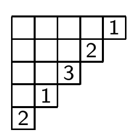
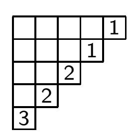
The fact that y is in S^α, which we already checked, is equivalent to the fact that each row of the tableau corresponding to y is in increasing order. Thus, A and B cannot be in the same row, since a and b would occur in the correct order in the tableau for x, and therefore out of order in the tableau for y.

We now rule out the possibility that A and B are adjacent in the same column. We can write $x = (x(a)\ x(b))y$. The effect of left-multiplying y by $(x(a)\ x(b))$ can be described as removing the simple reflection which we added to obtain y from x. Since, by the construction of y, we know that the words for x and y agree up to that simple reflection, $(x(a)\ x(b))z$ can also be obtained from z by removing the same simple reflection. The tableau corresponding to $(x(a)\ x(b))z$ can be obtained from that of z by swapping the entries in A and B. The fact that this decreases the length of z means that A contains the larger value and B contains the lower value in z. Since, in the case that we are considering, A and B are vertically adjacent, this contradicts our assumption that $z \in L$, which would imply that the tableau for z is standard.

This shows that y is lattice, as desired, which completes the proof. $\square$

Remark: This proof for the implication $Z <_{\mathsf{dom}} X \implies Z <_{\mathsf{box}} X$ is constructive as it exhibits the first box move: Let $s_{i_1} \cdots s_{i_p}$ be a reduced expression for $\pi(Z)$. Write $\pi(X)$ as a subword $s_{i_1} \cdots \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_q}} \cdots s_{i_p}$ such that j_1 is maximal. Then $s_{i_1} \cdots s_{i_{j_1-1}} s_{i_{j_1}} s_{i_{j_1-1}} \cdots s_{i_1} = (a, b)$ is a transposition and $(a, b)\,\pi(X) = \pi(Y)$ defines an LR-filling Y which satisfies $Z \leq_{\mathsf{dom}} Y$ and $Y <_{\mathsf{box}} X$.

Example: Consider

$Z:$  , $X:$ 

We have $\pi(Z) = 13524$, and $\pi(X) = 12345$. Let's say we want to move down from X. We write $\pi(Z) = (45)(23)(34) = s_4 s_2 s_3$. $\pi(X) = e$, so the subword of $s_4 s_2 s_3$ corresponding to e is the empty subword. We add the leftmost reflection back in, so that is $(a\ b) = s_4$. This gives us the permutation $\pi(Y) = 12354$, which corresponds to the filling $Y = 11232$, which is indeed lattice. To get the next step up the chain, we go to the subword $s_4 s_2 = 13254$

of z which comes from the filling $\hat{Y} = 12132$ with $\pi(\hat{Y}) = 13254$, which is again lattice. Here we apply the transposition $(a\ b) = s_4 s_2 s_4 = s_2$ to $\pi(Y)$. In the final step we obtain Z by applying the transposition $(a\ b) = s_4 s_2 s_3 s_2 s_4 = (2\ 5)$ to $\pi(\hat{Y})$.

5. The second proof of the main result and an algorithmic approach

Let X, Z be LR-fillings of type (α, β, γ), where $\beta \setminus \gamma$ is a rook strip. By Theorem 1.1, $Z \leq_{\text{dom}} X$ implies $Z \leq_{\text{box}} X$. In this section we present a simple and explicit procedure for finding box moves that transform X into Z. We assume that $\beta \setminus \gamma$ is a rook strip.

Notation: For an LR-filling X we denote by $\omega(X)$ the list of entries when reading columns from the bottom up, starting with the leftmost column and moving right. Clearly, X is determined uniquely by its shape and by the list of entries. By $\omega(X)_{\leq c}$ we denote the initial segment of $\omega(X)$ consisting of the entries in the first c columns.

For a filling X of type (α, β, γ), we denote by $e_c = \gamma'_1 + \cdots + \gamma'_c$ the number of empty boxes in the first c columns, and by $\#\omega(X)^{\leq u}_{\leq c}$ the number of entries at most u in $\omega(X)_{\leq c}$.

Lemma 5.1. *Suppose X, Z are LR-fillings of the same type such that $Z <_{\mathsf{dom}} X$ holds. Then $\omega(Z) < \omega(X)$ in the lexicographical order.*

Proof. Let $k \in \mathbb{N}$ be such that for $1 \leq i < k$ the entries $\omega(X)_i = \omega(Z)_i$ are equal, and $a = \omega(X)_k \neq \omega(Z)_k = c$. We need to show that $c < a$. Suppose the LR-fillings are given by partition sequences $X = (\gamma^{(i)})$, $Z = (\delta^{(i)})$. Since $\omega(Z)_k = c$, we have $(\delta^{(c)})'_k = \beta'_k$ and $(\delta^{(c-1)})'_k = \beta'_k - 1 = \gamma'_k$. Similarly, $(\gamma^{(a)})'_k = \beta'_k$ and $(\gamma^{(a-1)})'_k = \beta'_k - 1 = \gamma'_k$. Moreover since $a \neq c$, we have $(\delta^{(a-1)})'_k = (\delta^{(a)})'_k$.

Since $Z <_{\mathsf{dom}} X$, it follows from the definition that $\delta^{(a)} \leq_{\mathsf{dom}} \gamma^{(a)}$ holds, hence $\sum_{i\leq k}(\gamma^{(a)})'_i \leq \sum_{i\leq k}(\delta^{(a)})'_i$. As $\sum_{i<k}(\gamma^{(a)})'_i = \sum_{i<k}(\delta^{(a)})'_i$, we obtain $\beta'_k = (\gamma^{(a)})'_k \leq (\delta^{(a)})'_k$, hence $(\gamma^{(a)})'_k = (\delta^{(a)})'_k$. This implies that $c \leq a$. $\square$

Lemma 5.2. *$Z \leq_{\mathsf{dom}} X$ if and only if $\#\omega(Z)^{\leq u}_{\leq c} \geq \#\omega(X)^{\leq u}_{\leq c}$ holds for all u, c.*

Proof. First note that for an LR-filling $X = [\gamma^{(i)}]$, and for $u, c \in \mathbb{N}$, the number of boxes at most u in the first c columns is $\sum_{i\le c}(\gamma^{(u)})'_i = e_c + \#\omega(X)^{\le u}_{\le c}$.

The following statements are equivalent.

- $Z \le_{\mathsf{dom}} X$
- $\delta^{(u)} \le_{\mathsf{dom}} \gamma^{(u)}$, for all u
- $\sum_{i\le c}(\delta^{(u)})'_i \ge \sum_{i\le c}(\gamma^{(u)})'_i$, for all u, c
- $\#\omega(Z)^{\le u}_{\le c} \ge \#\omega(X)^{\le u}_{\le c}$, for all u, c.

□

Proof of Theorem 1.1. We have already seen that the box relation implies the dominance relation for LR-fillings.

For the converse, assume that X, Z are LR-fillings of the same type (α, β, γ) such that $Z <_{\mathsf{dom}} X$. We construct an LR-filling Y of type (α, β, γ) such that

$$Z \le_{\mathsf{dom}} Y \qquad \text{and} \qquad Y <_{\mathsf{box}} X.$$

Denote the word of X as $\omega(X) = (\omega_1, \dots, \omega_n)$. Let k be such that $\omega(X)$ and $\omega(Z)$ agree for the first $k-1$ letters and that $\omega_k = \omega(X)_k$ differs from $a = \omega(Z)_k$. By Lemma 5.1, $a < \omega_k$.

Suppose the first occurrence of a in $(\omega_{k+1}, \dots, \omega_n)$ is at position m, so $a = \omega_m$ with $m \in \{k+1, \dots n\}$ as small as possible. Let $b = \min\{\omega_i > a : k \le i < m\}$; here $b = \omega_k$ is allowed. Choose an arbitrary ℓ such that $b = \omega_\ell$ and $k \le \ell < m$.

We can now define Y as the filling obtained from X by exchanging b in column ℓ with a in column m. So $Y <_{\mathsf{box}} X$.

To verify that Y is an LR-filling, we need to check the lattice permutation property. Let q be a column; we may assume that $\ell < q \le m$. The interesting entries are b and $a+1$. We first deal with entry b. If $b > a+1$, then by the choice of b, all entries $b-1$ on the right hand side of column q actually are on the right hand side of column m. If $b = a+1$ the result follows by the choice of m, since Z is an LR-filling which agrees with Y in $(\omega_1, \dots, \omega_{k-1})$. It remains to deal with entry $a+1$. Note that since $\ell < q \le m$, this case is only possible if $b = a+1$ and has been considered above.

Next we show that $Z \leq_{\mathsf{dom}} Y$. Using Lemma 5.2, we show that the matrix

$$(\#\omega(Z)^{\leq u}_{\leq q} - \#\omega(Y)^{\leq u}_{\leq q})_{q,u}$$

has only nonnegative entries. For this, we compare the following two matrices ("$>$" stands for a positive, "$\geq$" for a nonnegative entry).

$$(\#\omega(Y)^{\leq u}_{\leq q} - \#\omega(X)^{\leq u}_{\leq q})_{q,u} : \qquad \begin{array}{c|ccccccccccc}
u \backslash q & 1 & & k-1 & k & & \ell-1 & \ell & & m-1 & & n \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
b-1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_k-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

The entries 1 arise since the boxes a and b in columns m and ℓ have been exchanged.

$$(\#\omega(Z)^{\leq u}_{\leq q} - \#\omega(X)^{\leq u}_{\leq q})_{q,u} : \qquad \begin{array}{c|ccccccccccc}
u \backslash q & 1 & & k-1 & k & & \ell-1 & \ell & & m-1 & & n \\
\hline
1 & 0 & 0 & 0 & 0 & \geq & \geq & \geq & \geq & \geq & \geq & 0 \\
 & 0 & 0 & 0 & 0 & \geq & \geq & \geq & \geq & \geq & \geq & 0 \\
a & 0 & 0 & 0 & \mathbf{1} & > & > & > & > & > & \geq & 0 \\
 & 0 & 0 & 0 & \mathbf{1} & > & > & > & > & > & \geq & 0 \\
b-1 & 0 & 0 & 0 & \mathbf{1} & > & > & > & > & > & \geq & 0 \\
b & 0 & 0 & 0 & \mathbf{1} & \geq & \geq & \geq & \geq & \geq & \geq & 0 \\
 & 0 & 0 & 0 & \mathbf{1} & \geq & \geq & \geq & \geq & \geq & \geq & 0 \\
a_k-1 & 0 & 0 & 0 & \mathbf{1} & \geq & \geq & \geq & \geq & \geq & \geq & 0 \\
a_k & 0 & 0 & 0 & 0 & \geq & \geq & \geq & \geq & \geq & \geq & 0 \\
 & 0 & 0 & 0 & 0 & \geq & \geq & \geq & \geq & \geq & \geq & 0
\end{array}$$

According to Lemma 5.2, since $Z <_{\mathsf{dom}} X$, all entries in the matrix are nonnegative. Since X and Z have the same first $k-1$ columns, all corresponding entries are 0. The different entries in the k-th columns of X and Z, respectively, give rise to the 1's. Since there are no entries $a, \ldots, b-1$ in columns $k, \ldots, m-1$ in X, the matrix entries there are strictly positive.

We see that the difference of the two matrices pictured is nonnegative in every component, hence the matrix $(\#\omega(Z)^{\leq u}_{\leq q} - \#\omega(Y)^{\leq u}_{\leq q})_{q,u}$ has only nonnegative entries. This shows $Z \leq_{\mathsf{dom}} Y$.

Finally, we repeat the process of splitting off box moves from Y until we reach Z. This process terminates after finitely many steps since the sum of the entries in the matrix $(\#\omega(Z)^{\leq u}_{\leq q} - \#\omega(Y)^{\leq u}_{\leq q})_{q,u}$ is nonnegative and decreases with each box move. □

Algorithm: The proof presented above gives the following algorithm.

Input: LR-fillings X, Z of type (α, β, γ) such that $\beta \setminus \gamma$ is a rook strip and $Z <_{\mathsf{dom}} X$.

Output: LR-filling Y of type (α, β, γ) such that $Z \leq_{\mathsf{dom}} Y$ and $Y <_{\mathsf{box}} X$.

Step 1. Find the smallest k such that $\omega(Z)_k \neq \omega(X)_k$ and put $a = \omega(Z)_k$.

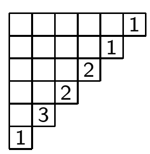
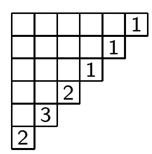
Step 2. Choose the minimal $m \geq k+1$ such that $a = \omega(X)_m$.

Step 3. Let $b = \min\{\omega(X)_i > a : k \leq i < m\}$.

Step 4. Choose an arbitrary $k \leq \ell < m$ such that $b = \omega(X)_\ell$.

Step 5. Define Y such that $\omega(Y)_i = \omega(X)_i$, for $i \neq \ell, m$, and $\omega(Y)_\ell = a$, $\omega(Y)_m = b$.

Example: Let $\beta = (6,5,4,3,2,1)$, $\gamma = (5,4,3,2,1)$ and $\alpha = (3,2,1)$. Consider two LR-fillings Z and X of type (α, β, γ) such that $\omega(Z) = (1,3,2,2,1,1)$ and $\omega(X) = (2,3,2,1,1,1)$. Clearly, $\beta \setminus \gamma$ is a rook strip and $Z <_{\mathsf{dom}} X$.

$Z:$  $<_{\mathsf{dom}}$ $X:$ 

We apply the algorithm. Note that $k = 1$, $a = 1$, $m = 4$, and $b = 2$. Now we can choose $\ell = 1$ or $\ell = 3$. If $\ell = 1$, then $Y = Z$. If $\ell = 3$, we get $\omega(Y) = (2,3,1,2,1,1)$. It is easy to see that $Z <_{\mathsf{dom}} Y$ and we can continue.

We conclude this section with a remark on the runtime of this and the previous algorithm.

Recall that the second algorithm may depend on choices. Here we focus on the version of the algorithm where in Step 4, the ℓ is always chosen as large as possible. Note that this particular choice will make the algorithm as slow as possible. (The first algorithm also depends on choices. We conjecture in Section 7 that for suitable choices, the two algorithms do in fact produce the same sequence of box moves.)

Lemma 5.3. *The algorithms in Section 4 and in Section 5 take the same number of steps. More precisely, for given $Z <_{\mathsf{dom}} X$, the number of steps*

is the distance of $\pi(Z)$ from $\pi(X)$ in the Bruhat order, that is, there are $\ell(\pi(Z)) - \ell(\pi(X))$ steps.

Proof. The algorithm in Section 4 produces a sequence of coverings from $\pi(X)$ to $\pi(Z)$, hence the number of steps is as specified (this is the Chain Property of the Bruhat order, see [1, Theorem 2.2.6]).

It remains to show that for Y as constructed in the algorithm in Section 5, $\pi(Y)$ is a covering for $\pi(X)$. We write $\pi(X) = (x_1, \ldots, x_n)$ and show that $\pi(Y) = \pi(X) \cdot \tau$ where $\tau = (u\ v)$ is a transposition with the property that $x_u < x_v$ and there does not exist any w such that $u < w < v$ and $x_u < x_w < x_v$. Then $\pi(Y)$ is a covering for $\pi(X)$ by [1, Lemma 2.1.4].

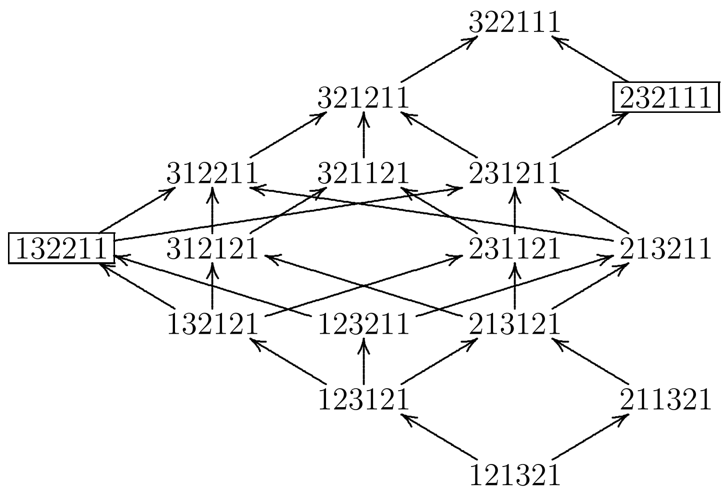
Note that $\pi(X)$ is the standardization of the word obtained by reading $\omega(X) = (\omega_1, \ldots, \omega_n)$ backwards, see Section 4. $\omega(Y)$ is obtained by interchanging $a = \omega_m$ in position m and $b = \min\{\omega_i > a : k \leq i < m\}$ in position ℓ where $\ell \in \{k, \ldots, m-1\}$ is as large as possible. Recall that there is no entry a in positions $k, \ldots, m-1$. Hence there is no r with $\ell < r < m$ and $b = \omega_\ell \geq \omega_r \geq \omega_m = a$. Putting $u = n+1-m$, $v = n+1-\ell$, it follows that there is no w such that $u < w < v$ and $x_u < x_w < x_v$. $\square$

6. Combinatorial properties of the order $\leq_{\mathsf{box}}$

In this section we study combinatorial properties of the posets $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ and $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{dom}})$, mainly in the case where $\beta \setminus \gamma$ is a rook strip.

6.1. An example

Consider the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$, where $\beta = (6,5,4,3,2,1)$, $\gamma = (5,4,3,2,1)$ and $\alpha = (3,2,1)$. All LR-fillings of this type have entries: $1,1,1,2,2,3$. The Hasse diagram of $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ is the following (instead of X we write $\omega(X)$):



Consider the LR-fillings in frames. One is obtained from the other by a single box move, but there is no chain of neighboring moves, i.e. moves that exchange neighbors. Moreover note that:

- $\beta \setminus \gamma$ is a rook strip;
- in this poset there exists exactly one maximal and exactly one minimal element;
- all saturated chains have the same length;
- this poset is not a lattice;

6.2. Maximal and minimal elements

We have seen in [5, Proposition 5.5] that the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{dom}})$ has a unique maximal and a unique minimal element in the case where all parts of α are at most 2. We show that this statement also holds true if $\beta \setminus \gamma$ is a rook strip.

Lemma 6.1. *Assume that $\beta \setminus \gamma$ is a rook strip. In the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ there exists:*

1. *exactly one maximal element: the LR-filling X such that the coefficients in $\omega(X)$ are in non-increasing order,*

2. *exactly one minimal element: the LR-filling X such that $\omega(X)$ has the form $(p_1^{k_1}, p_2^{k_2}, \dots, p_n^{k_n})$, for some $k_1, \dots, k_n$, where $p_i = (i, i-1, \dots, 1)$ and $p^k = (p, p, \dots, p)$ (k times).*

Proof. 1. Let X be such that the coefficients in $\omega(X) = (\omega_1, \dots, \omega_n)$ are not in non-increasing order. Then there exists j such that $\omega_{j-1} < \omega_j$. Put

$$\omega(\widetilde{X}) = (\omega_1, \dots, \omega_{j-2}, \omega_j, \omega_{j-1}, \omega_{j+1}, \dots, \omega_n);$$

then $\widetilde{X}$ is an LR-filling with $X <_{\mathsf{box}} \widetilde{X}$, so X is not maximal.

2. Note that the element of the required form $(p_1^{k_1}, p_2^{k_2}, \dots, p_n^{k_n})$ is created as follows: Starting from the right hand side we always take the largest possible entry.

Let X be such that $\omega(X)$ does not have the required form. Choose i maximal with the property that ω_i is not the largest possible entry. (It follows from the lattice permutation property that $\omega_i = 1$.) Choose $j < i$ maximal with the property that ω_j is the largest possible entry that can be in the place i. It follows that $j < i$ and $\omega_j > \omega_k$ for all $k = j+1, \dots, i$, again by the lattice permutation property. We can interchange ω_j and ω_{j+1} and get an LR-filling Z such that $Z <_{\mathsf{box}} X$. This finishes the proof. □

In terms of the motivation in Section 3, the lemma has the following interpretation in terms of the varieties of short exact sequences of linear operators which correspond to the tableaux.

Corollary 6.2. *Given partitions α, β, γ such that $\beta \setminus \gamma$ is a rook strip, the variety*

$$\mathbb{V}_{\alpha,\gamma}^{\beta} = \bigcup_{X \in \mathcal{T}_{\alpha,\gamma}^{\beta}} \mathbb{V}_X$$

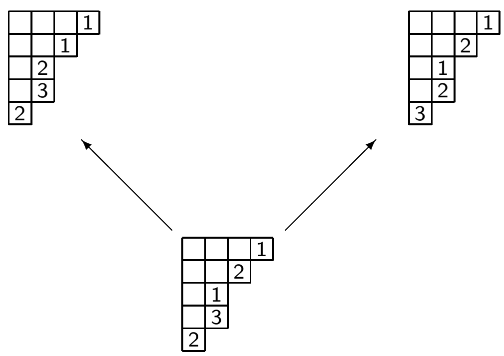
has a unique open component $\mathbb{V}_O$ and a unique closed component $\mathbb{V}_C$. The open component $\mathbb{V}_O$ and the closed component $\mathbb{V}_C$ are given by the LR-tableaux O and C, respectively, which correspond to the unique minimal and the unique maximal element in the poset $(\mathcal{T}_{\alpha,\gamma}^{\beta}, \leq_{\mathsf{dom}})$.

Proof. The following statements are equivalent for an LR-tableau $O \in \mathcal{T}_{\alpha,\gamma}^{\beta}$: (1) O is a minimal element in the poset $(\mathcal{T}_{\alpha,\gamma}^{\beta}, \leq_{\mathsf{dom}})$; (2) O is a minimal element in the poset $(\mathcal{T}_{\alpha,\gamma}^{\beta}, \leq_{\mathsf{boundary}})$; (3) there is no $X \in \mathcal{T}_{\alpha,\gamma}^{\beta} \setminus \{O\}$ such that $\mathbb{V}_O \cap \overline{\mathbb{V}}_X \neq \emptyset$; (4) the union $\cup_{X \neq O} \mathbb{V}_X$ is closed in $\mathbb{V}_{\alpha,\gamma}^{\beta}$; (5) the variety $\mathbb{V}_O$ is open in $\mathbb{V}_{\alpha,\gamma}^{\beta}$. Hence, since the poset $(\mathcal{T}_{\alpha,\gamma}^{\beta}, \leq_{\mathsf{dom}})$ has O as the unique

minimal element, the component $\mathbb{V}_O$ is the unique open component in the decomposition in the lemma.

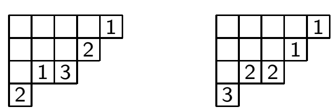
Similarly, the following statements are equivalent for $C \in \mathcal{T}^{\beta}_{\alpha,\gamma}$: (1) C is a maximal element in the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{dom}})$; (2) C is a maximal element in the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{boundary}})$; (3) there is no $X \in \mathcal{T}^{\beta}_{\alpha,\gamma} \setminus \{C\}$ such that $\mathbb{V}_X \cap \overline{\mathbb{V}}_C \neq \emptyset$; (4) the variety $\mathbb{V}_C$ is closed in $\mathbb{V}^{\beta}_{\alpha,\gamma}$. The uniqueness of the maximal element C in $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{dom}})$ implies that $\mathbb{V}_C$ is the unique closed component in the decomposition. $\square$

Example: 1. The first example shows that the condition that $\beta \setminus \gamma$ be a horizontal strip is needed for the uniqueness of the maximal element in $\mathcal{T}^{\beta}_{\alpha,\gamma}$. Consider the partition triple $\beta = (4,3,2,2,1)$, $\gamma = (3,2,1,1)$ and $\alpha = (2,2,1)$. The Hasse diagram of the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{dom}})$ has the following shape:

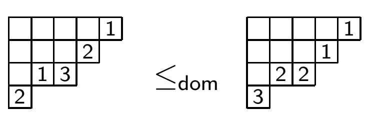


2. The second example shows that in Theorem 1.1, the condition that $\beta\setminus\gamma$ be a vertical strip is necessary. We also see that for horizontal strips, the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ may have several minimal and several maximal elements.

   Let $\beta = (5,4,3,1)$, $\gamma = (4,3,2,1)$ and $\alpha = (2,2,1)$. There are two LR-fillings of type (α, β, γ):



   They are incomparable in $\leq_{\mathsf{box}}$ relation, but



6.3. Saturated chains

Theorem 6.3. *If $\beta \setminus \gamma$ is a rook strip, then all saturated chains in the poset $(\mathcal{T}^{\beta}_{\alpha,\gamma}, \leq_{\mathsf{box}})$ have the same length.*

Proof. Since covers in the Bruhat order increase the number of inversions by 1, the theorem follows from Theorem 1.1, Proposition 4.1, Lemma 4.5 and Proposition 4.6. □

7. Open Problem

We want to compare the two algorithms presented above. Both algorithms take as input two LR-fillings X, Z of the same type which is a rook strip and such that $Z <_{\mathsf{dom}} X$. They produce as output LR-fillings Y_1 (the first algorithm) and Y_2 (the second algorithm) such that $Z \leq_{\mathsf{dom}} Y_i$ and $Y_i <_{\mathsf{box}} X$, for $i = 1, 2$.

Both algorithms depend on choices. In this section, we describe a way to make the choices so that (conjecturally) the two algorithms produce the same result. The first algorithm depends on the choice of a factorization of z as a product of adjacent transpositions. To produce the reduced expression for $z \in S_n$, a version of the "bubble sort" algorithm is used, as in [1, Example 3.4.3], but we are working on the left instead of the right side (or, equivalently, we apply the bubble sort algorithm to z^{-1}). Namely, starting from the permutation z in mapping notation, and assuming that the entry $u = z(n)$ is in position n, increase this entry until it equals n by successively exchanging u and $u+1$, $u+1$ and $u+2$, etc. This gives the permutation $s_{n-1} \cdots s_{u+1} s_u \cdot z = z_{(n)}^{-1} \cdot z$. Next, move the entry $n-1$ to the second position from the right by successively exchanging u' with $u'+1$, $u'+1$ with $u'+2$, etc. where u' is the entry in position $n-1$ in $z_{(n)}^{-1} \cdot z$. This gives the permutation $s_{n-2} \cdots s_{u'+1} s_{u'} \cdot z_{(n)}^{-1} z = z_{(n-1)}^{-1} z_{(n)}^{-1} z$. Then continue with the entry $n-2$, and so on. We obtain $e = z_{(2)}^{-1} z_{(3)}^{-1} \cdots z_{(n)}^{-1} \cdot z$.

In the second algorithm, the position ℓ of the second entry may not be unique. We assume that ℓ is always chosen as large as possible.

Conjecture 7.1. *Let X, Z, Y_1, Y_2 be as above. If we apply both algorithms with the choices described above, then $Y_1 = Y_2$.*

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